PERFECT PRISMS

Recall the following.

Definition 0.1. *A δ-pair (A, I) is a* ***prism*** *if*

(i) I is a Cartier divisor
(ii) A is derived (p, I)-complete, and
(iii) $p \in I + \phi(I)A$.

We say that (A, I) is a perfect prism if the Frobenius lift $\phi : A \to A$ is an automorphism.

Theorem 0.2. *There is an equivalence of categories*

$$\{p - \text{complete perfect } \delta\text{-rings}\} \cong \{\text{perfect } \mathbb{F}_p\text{-algebras}\}$$
$$A \mapsto A/p$$
$$W(B) \leftarrow B$$

1. Basic properties

Let (A, I) be a perfect prism.

(i) We saw last time that $\phi(I)A$ is principal, generated by a distinguished element. Since ϕ is now an isomorphism, this implies that I is principal, generated by a distinguished element.

(ii) The second point in the definition also simplifies:

Lemma 1.1. *If R is a perfect $\mathbb{F}_p$-algebra and $f \in R$, then $R[f^\infty] = R[f]$.*

Proof. If $f^r x = 0$ for some $r \geq 0$, then $f^{p^n} x^{p^n} = 0$ for some n, so $fx = 0$ as R is perfect. □

Note that the same argument shows that $R[f^\infty] = R[f^{1/p^m}]$ for any $m \geq 0$. We claim that this implies that A is in fact classically (p, I)-complete.
Proof: A/p is perfect and derived I-complete. But by the above, it has bounded d-torsion, so derived I-completeness implies classical I-completeness. Dévissage implies then that A/p^n is classically I-complete (as a repeated extension of copies of A/p). As A is p-torsionfree, derived p-completeness implies classical p-completeness, so that $A \cong \varprojlim A/p^n$. Thus $A \cong \varprojlim(\varprojlim A/(p^n, d^m))$ is (p, I)-complete.

(iii) It folllows from the Theorem above that $A \cong W(A/p)$, and we saw last time what distinguished elements look like in this case: $d = \sum [d_i]p^i$ with $d_1 \in (A/p)^\times$.

(iv) A standard δ-ring calculation shows that $A/I[p^\infty] = A/I[p]$, so in particular any perfect prism is **bounded**.
(For the interested: Suppose that f is killed by p^2, so $p^2 f = gd$ for some $g \in A$. Then $\delta(gd) \in pA$, so

$$\delta(d)g^p\phi(g) + \delta(g)\phi(gd) = \phi(g)\delta(gd) \in pA.$$

Since $gd \in p^2A$, so is $\phi(gd)$, so $\delta(d)$ being a unit implies $g^p\phi(g) \in pA$. Then $g^{2p} \in pA$, but A/p is reduced. Thus $g \in pA$, so f is already killed by p.)

2. Perfectoid rings

Given a prism (A, I), prismatic cohomology yields a way to study (A/I)-schemes in terms of prisms over A (e.g. in the crystalline case, this means that we lift e.g. from $\mathbb{F}_p$-schemes to structures over $\mathbb{Z}_p$). In the case of perfect prisms, this going back-and-forth works... well, perfectly.

Definition 2.1. *A commutative ring is* ***perfectoid*** *if it is of the form A/I for some perfect prism (A, I).*

Theorem 2.2. *There is an equivalence of categories*

$$\{\textit{perfect prisms}\} \cong \{\textit{perfectoid rings}\}$$
$$A \mapsto A/I.$$

Note that this is a generalization of Theorem 0.2, which covers the crystalline case $I = (p)$.

Proof. (Sketch). Given a perfectoid ring $R = A/I$, we need to recover (A, I) in a functorial way. Write $I = (d)$ for a distinguished element d, as before.
Define $R^\flat = \varprojlim_\phi R/p$, the perfection of R/p. We claim that $A \cong W(R^\flat)$. In fact, $R/p = A/(p, d)$, and $\phi^n : R/p \to R/p$ corresponds to the natural map $A/(p, d^{p^n}) \to A/(p, d)$, as A/p is perfect.
Thus $R^\flat \cong \varprojlim A/(p, d^{p^n})$ is the d-adic completion of A/p. But A/p is already d-adically complete, so $R^\flat \cong A/p$, and $A \cong W(A/p) \cong W(R^\flat)$. This proves the claim.
By construction, we have a surjective map $R^\flat \to R/p$, which is simply projecting on the first factor. Since $\mathbb{L}_{A/p/\mathbb{F}_p} = 0$, deformation theory tells us that this lifts uniquely to a map $\theta : A = W(R^\flat) \to R$. By derived Nakayama, θ is surjective. By uniqueness of the lift, θ is the natural quotient map $A \to R = A/I$, so we recover I as $\ker\theta$. □

Remark. *We call $R^\flat$ the* ***tilt*** *of R and write $A_{\mathrm{inf}}(R) := W(R^\flat)$. This generalizes Fontaine's original period ring $A_{\mathrm{inf}} = A_{\mathrm{inf}}(\mathfrak{o}_{\mathbb{C}_p})$. Together with the crystalline case (i.e. perfect $\mathbb{F}_p$-algebras), something like $\mathfrak{o}_{\mathbb{C}_p}$ is a key example of a perfectoid ring to keep in mind.*
Note that for any p-adically complete ring R, we can define $A_{\mathrm{inf}}(R) = W(R^\flat)$, together with a surjective map $\theta : A_{\mathrm{inf}}(R) \to R$.

While our definition of perfectoid rings seems non-standard, one can show that it agrees with what is called 'integral pefectoid' in earlier work by Bhatt–Scholze:

Proposition 2.3. *A commutative ring R is perfectoid if and only if all of the following hold:*

(i) *R is classically p-adically complete.*
(ii) *The Frobenius $\phi : R/p \to R/p$ is surjective.*
(iii) *$\theta : A_{\mathrm{inf}}(R) \to R$ has principal kernel.*
(iv) *there exists $\varpi \in R$ such that $\varpi^p = pu$ for some $u \in R^\times$.*

If R is p-torsionfree, (iii) can be replaced by: if $x \in R[1/p]$ with $x^p \in R$, then $x \in R$ (i.e. R is p-normal).

We omit the proof here and only point out that perfectoid rings clearly satisfy (i), (ii), and (iii). For (iv), write $d = \sum [d_i]p^i$. Since $[d_1]$ is a unit, we have $d = [d_0] - p \cdot u$ for a unit $u \in R^\times$, so we set $\varpi = \theta([d_0^{1/p}])$.

Remark. *Since $R/p \cong A/(p,d) = A/(p,[d_0])$ and A/p is perfect, we see that the kernel of the Frobenius $R/p \to R/p$ is then generated by the image of ϖ in R/p.*

We also note that ϖ admits a system of p-power roots by taking $\varpi^{1/p^n} = \theta([d_0^{1/p^n}])$.

Lemma 2.4. *Let R be a perfectoid ring. Then $R[p^\infty] = R[p] = R[\sqrt{pR}]$, and $\sqrt{pR} = \cup(\varpi^{1/p^n})$.*

Proof. We have already shown the first equality.
For the equality $\sqrt{pR} = \cup(\varpi^{1/p^n}$, note that $(p) \subseteq \cup(\varpi^{1/p^n}) \subseteq \sqrt{pR}$. As $R^\flat$ is perfect, so is $R/\cup(\varpi^{1/p^n}) \cong R^\flat/\cup(d_0^{1/p^n})$, in particular it is reduced. Thus $\sqrt{pR} = \cup(\varpi^{1/p^n})$.
Lastly, we have

$$R[p] = A/d[p] = A/p[d] = R^\flat[d],$$

where the second equality is the so-called torsion-exchange lemma: as both p and d are non-zero-divisors, both expression can be computed as the first homology of a suitable Koszul complex. Note that in $R^\flat = A/p$, d is equal to d_0. By Lemma 1.1, $R^\flat[d_0^\infty] = R^\flat[d_0^{1/p^n}]$ for any n, so any element in $R[p]$ is annihilated by ϖ^{1/p^n} for all n. As we have already shown that $\sqrt{pR} = \cup(\varpi^{1/p^n}$, this concludes the proof that $R[p] = R[\sqrt{pR}]$. □

We call $\overline{R} = R/\sqrt{pR}$ the **special fibre** of R. The above shows that this is a perfect $\mathbb{F}_p$-algebra.

3. A structure theorem

The examples of perfectoid rings we have seen so far were either p-torsion (perfect $\mathbb{F}_p$-algebras) or p-torsionfree (like $\mathfrak{o}_{\mathbb{C}_p}$). It turns out that all perfectoid rings are built of these two classes.

Proposition 3.1. *Let R be a perfectoid ring, $\overline{R} = R/\sqrt{pR}$, $S = R/R[\sqrt{pR}]$, and $\overline{S} = S/\sqrt{pS}$.*
Then $\overline{R}$, S and $\overline{S}$ are also perfecoid, and the natural diagram

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ \overline{R} & \longrightarrow & \overline{S} \end{array}$$

is both a pullback and a pushout square (in the category of commutative rings). Moreover,

(i) S is p-torsionfree.
(ii) $\sqrt{pR}$ maps isomorphically to $\sqrt{pS}$.
(iii) $R[\sqrt{pR}]$ maps isomorphically to $\ker(\overline{R} \to \overline{S})$, and thus $x \mapsto x^p$ is bijective on $R[\sqrt{pR}]$ (as $\overline{R}$ is perfect).

Proof. Note that (i) is immediate from the previous lemma.
We will first show that the given diagram is a pullback diagram.
We write $A = A_{\mathrm{inf}}(R)$, $R = A/(d)$ for a distinguished element d with $d = [d_0] - p \cdot u$.
Let $I = \cup(d_0^{1/p^n}) \subset R^\flat$, and let $J = R^\flat[I]$.

Step 1: The diagram

$$\begin{array}{ccc} R^\flat & \longrightarrow & R^\flat/J \\ \downarrow & & \downarrow \\ R^\flat/I & \longrightarrow & R^\flat/(I+J) \end{array}$$

is a pullback diagram, and J and $I + J$ radical ideals.
The ideals in question are radical by Lemma 1.1. To show that the diagram of quotient maps is a pullback diagram, it remains to show that $I \cap J = \{0\}$. But $R^\flat$ is perfect, so in particular reduced, and if $x \in I$ and $x \in R[I]$, then $x^2 = 0$, so $x = 0$.

Step 2: As the ideals in the previous step were radical, the respective quotients are all still perfect. The diagram

$$\begin{array}{ccc} W(R^\flat) & \longrightarrow & W(R^\flat/J) \\ \downarrow & & \downarrow \\ W(R^\flat/I) & \longrightarrow & W(R^\flat/(I+J)) \end{array}$$

is then also a pullback square by dévissage, as the Witt rings are p-torsionfree and p-adically complete.

Step 3: In all four rings, d is a non-zero divisor, giving us short exact sequences e.g.

$$0 \longrightarrow W(R^\flat) \xrightarrow{d} W(R^\flat) \longrightarrow R \to 0.$$

A careful diagram chase (using that all maps in our previous square were surjective) yields that

$$\begin{array}{ccc} R & \longrightarrow & W(R^\flat/J)/d \\ \downarrow & & \downarrow \\ W(R^\flat/I)/d & \longrightarrow & W(R^\flat/(I+J))/d \end{array}$$

is a pullback square.

It thus remains to show that this square can be identified with the one in the Proposition.
Note that in $W(R^\flat/I)$, $(d) = (p)$, since $d_0 \in I$. Thus

$$W(R^\flat/I)/d = R^\flat/I = R^\flat/\cup(d_0^{1/p^n}) = R/\sqrt{pR},$$

so $W(R^\flat/I)/d \cong \overline{R}$, as required.
Write $S' = W(R^\flat/J)/d$. The same argument as above shows that

$$W(R^\flat/(I+J))/d \cong \overline{S'},$$

so it remains to show that $S \cong S'$.
By torsion-exchange $S'[p] = W(R^\flat/J)/p[d]$. The latter is zero, as $d = d_0$ is a non-zero divisor in R^b/J (use again Lemma 1.1). Thus S' is p-torsionfree, and the map $R \to S'$ factors through S. It remains to show that the kernel K of $R \to S'$ is contained in $R[p^\infty]$. But since the square is a pullback, K embeds into $\overline{R}$, which is of characteristic p. Thus $S \cong S'$.

Thus our square is a pullback square, and we exhibited all rings as perfectoid rings. (ii) and (iii) follow directly from the above, which in turn implies that the square is also a pushout. □

Corollary 3.2. *Perfectoid rings are reduced.*

Proof. By above, wlog R is either p-torsionfree or perfect of characteristic p. The latter case is obvious, so assume R is p-torsionfree. Choose $\varpi \in R$ with $\varpi^p = pu$ as before. If $x \in R$ such that $x^p = 0$, we show inductively that $x \in \varpi^n R$: if $x = \varpi^n y$, then $\varpi^{pn} y^p = p^n u^n y^p = 0$, so $y^p = 0$ by p-torsionfreeness. As the kernel of the Frobenius $R/p \to R/p$ is generated by ϖ, this implies that $y \in \varpi R + pR = \varpi R$, so $x \in \varpi^{n+1} R$.
Now R is p-adically separated so if $x \in \varpi^n R$ for all n, then $x = 0$, as required. □